

# Ranked Fragmentations

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## Résumé

In this paper we define and study self-similar ranked fragmentations. We first show that any ranked fragmentation is the image of some partition-valued fragmentation, and that there is in fact a one-to-one correspondence between the laws of these two types of fragmentations. We then give an explicit construction of homogeneous ranked fragmentations in terms of Poisson point processes. Finally we use this construction and classical results on records of Poisson point processes to study the small-time behavior of a ranked fragmentation.

## 1 Introduction

Splitting models are meant to describe an object that falls apart. Applications are numerous and may be found in various fields such as physical chemistry (aerosols, phase separation, polymerization), mathematical population genetics or astronomy (we refer to [2] for a survey on applications and motivations).

This paper focuses on self-similar ranked fragmentation. For the sake of describing our results, let us just give some heuristic descriptions while precise definitions will be given in the next sections.

Imagine a unit-mass object that fragments as time runs. We only consider the ordered sequence of the fragments masses of this object so the state space is

$$\mathcal{S}^\downarrow := \{s = (s_1, s_2, \dots), s_1 \geq s_2 \geq \dots \geq 0, \sum_i s_i \leq 1\},$$

the situation where  $\sum_i s_i < 1$  corresponding to the fact that a part of the initial mass has been lost, i.e. the sum of the masses of the remaining fragments is less than the original total mass.

Let  $\lambda = (\lambda(t), t \geq 0)$  be a Markov process with values in  $\mathcal{S}^\downarrow$ . Call  $\lambda$  a self-similar ranked fragmentation if it fulfills the *scaling* and the *fragmentation* property.

The *scaling* property means that there exists a real number  $\alpha$ , called the index of self-similarity, such that if  $\mathbb{P}_r$  is the law of  $\lambda$  started from  $(r, 0, 0, \dots)$  then the distribution of  $(r\lambda(r^\alpha t), t \geq 0)$  under  $\mathbb{P}_1$  is  $\mathbb{P}_r$ .

The *fragmentation* property is a version of the branching property i.e. for any  $u, t \geq 0$ , for any  $s = (s_1, s_2, \dots) \in \mathcal{S}^\downarrow$ , conditionally on  $\lambda(u) = s$ ,  $\lambda(t + u)$  has the same distribution as the variable obtained by concatenating and ordering the sequences  $\lambda^{(1)}, \lambda^{(2)}, \dots$  where for each  $i$ ,  $\lambda^{(i)}$  has the distribution of  $\lambda(t)$  under  $\mathbb{P}_{s_i}$ .

Here is a simple prototype taken from Brennan and Durrett [8, 9] who consider the following model for polymer degradation : A particle of mass  $m$  splits with exponential rate  $m^\alpha$ ,  $\alpha \in \mathbb{R}^+$ , and gives rise to two particles of mass  $Vm$  and  $(1 - V)m$ , where  $V$  is a random variable with values in  $(0, 1)$  independent of the past. The new particles follow the same dynamic independently. The ordered sequence of the particles masses is a self-similar ranked fragmentation of index  $\alpha$ .

This example can be extended in two ways. First one can suppose that when a particle splits, it might give birth to any number of particles, possibly infinite, and not just two. Second, in the example of Brennan and Durrett, the splitting times are "discrete", the first time of splitting is almost surely strictly positive. It is natural to consider more generally the case where fragmentation may occur continuously. For instance this happens for the fragmentation process obtained by logging the continuous random tree of Aldous in [3].

In the existing literature, a somewhat different class of processes has been considered, the so-called *partition valued fragmentation*. Roughly speaking a partition fragmentation, say  $\Pi(t)$ , is a process that lives in the space of partitions of  $\mathbb{N}$ , such that for any  $0 < s \leq t$ ,  $\Pi(t)$  is a refinement of  $\Pi(s)$ . A way to construct such a fragmentation which makes clear the connection with the above particle model is the following : imagine an object  $E$  endowed with a unit mass measure  $\mu$  that falls apart as time runs, call *object fragmentation* the process  $F(t)$  with values in partitions of  $E$  that describes this fragmentation. Next, let  $(u_i)_{i \in \mathbb{N}}$  be a sequence of iid  $E$ -valued variables with

distribution  $\mu$  and for each  $t$  let  $\Pi_F(t)$  be the partition of  $\mathbb{N}$  such that for all  $i$  and  $j$  in  $\mathbb{N}$ ,  $i$  and  $j$  belong to the same block of  $\Pi(t)$  iff  $u_i$  and  $u_j$  are in the same fragment of  $E$  at time  $t$ . By the SLLN we can recover the mass of a fragment as the asymptotic frequency of the corresponding block. Then  $\Pi_F$  is a partition fragmentation.

Using partition fragmentations to construct ranked fragmentations is typical of the existing results. These constructions benefit from two important features : there is a clear genealogical structure, and partition fragmentations are characterized by an index of self-similarity  $\alpha$  and a so-called characteristic exchangeable measure, on which results concerning exchangeability can be usefully applied (see [1] for a survey on exchangeability).

However partition-valued fragmentations are perhaps less natural and could be less general than ranked fragmentations, precisely because we have endowed it with this extra genealogic structure. In other words it is not clear that an arbitrary ranked fragmentation can be studied through partition fragmentations.

In section 2 we show that it is in fact the case, and more precisely that for any ranked fragmentation  $\lambda$  we can associate a partition fragmentation  $\Pi$  such that the asymptotic frequencies of  $\Pi$  has same distribution as  $\lambda$ .

In the next section we use this equivalence between ranked and partition fragmentations to give a Poisson construction of homogeneous ranked fragmentation which is an analogue of that given in [5] for partition fragmentations. The difficulty comes from the fact that we can no longer use a genealogic structure, which played a crucial role in the partition case.

In section 4, this construction allows us to tackle the study of small time behavior of a ranked fragmentation. We show that the 2nd largest fragment, correctly renormalized, behaves as the record of the size of the particles detaching from the main fragment.

## 2 Definitions and first properties

### 2.1 Ranked Fragmentations

For each  $l$  in  $[0, 1]$  let  $P(l)$  be a probability on  $\{s \in \mathcal{S}^\downarrow : \sum_i s_i \leq l\}$  the space of all the possible fragmentations of  $l$ . Then for  $L = (l_1, l_2, \dots)$ , define  $P(L)$  as the distribution on  $\mathcal{S}^\downarrow$  of the concatenation and the decreasing rearrangement of independent  $\mathcal{S}^\downarrow$ -valued variables with respective law  $P(l_i)$ .

Call  $(P(L), L \in \mathcal{S}^\downarrow)$  a fragmentation kernel on  $\mathcal{S}^\downarrow$ . One says that the family  $(P(l), l \in [0, 1])$  generates  $(P(L), L \in \mathcal{S}^\downarrow)$ .

**Définition 1** *An  $\mathcal{S}^\downarrow$ -valued process  $\lambda(\cdot)$  is called a  $\mathcal{S}^\downarrow$ -fragmentation if it is a time-homogeneous Markov process such that*

1.  $\lambda$  is continuous in probability and starts from  $\lambda(0) = (1, 0, 0, \dots)$  a.s.
2. the transition semigroup  $(P_t(L))$  of  $\lambda$  is given by fragmentation kernels.

In words, at a given time  $t$ , each fragment of  $\lambda(t) = (\lambda_1(t), \lambda_2(t), \dots)$ , say  $\lambda_i(t)$ , gives rise to an independent fragmentation process which distribution only depends on the value  $\lambda_i(t)$ .  $\lambda$  is the concatenation and the reordering of all those processes.

For  $l \in [0, 1]$ , let  $g_l$  be the application from  $\mathcal{S}^\downarrow \rightarrow \mathcal{S}^\downarrow$  defined by

$$g_l : x = (x_1, x_2, \dots) \rightarrow (lx_1, lx_2, \dots).$$

**Définition 2** *The fragmentation  $\lambda$ , with transition kernels generated by the family  $(P_t(l); t \geq 0, l \in [0, 1])$  is said to be self-similar with index  $\alpha \in \mathbb{R}$  if (in the notations introduced above) for all  $l \in [0, 1]$  the distribution  $P_t(l)$  coincides with the image of  $P_{l^\alpha t}(1)$  by  $g_l$ .*

*When  $\alpha = 0$  the fragmentation is said to be homogeneous.*

$\mathcal{S}^\downarrow$  is endowed with the uniform distance. Note that for any  $s = (s_1, s_2, \dots) \in \mathcal{S}^\downarrow$  we must have for every  $k \in \mathbb{N}$ ,  $s_k \leq \frac{1}{k}$ , and thus the uniform and point-wise convergences are the same. In this setting we prove that a self-similar  $\mathcal{S}^\downarrow$ -fragmentation has the Feller property.

**Proposition 3** *(Feller property) The semi-group  $P_t$  of a self-similar ranked fragmentation of index  $\alpha$ , fulfills the Feller property. That is  $\forall t \geq 0$  the map*

$$L \rightarrow P_t(L)$$

*is continuous on  $\mathcal{S}^\downarrow$  and for each fixed  $L \in \mathcal{S}^\downarrow$ ,  $P_t(L)$  converge to the Dirac mass at  $L$  as  $t \rightarrow 0$ .*

**Proof.**

Consider a sequence  $(L_n, n \in \mathbb{N})$  in  $\mathcal{S}^\downarrow$  which converges to  $L \in \mathcal{S}^\downarrow$ . Note  $L_n = (l_1^{(n)}, l_2^{(n)}, \dots)$ , then for all  $k$ ,  $l_k^{(n)} \rightarrow l_k$ .

Let  $(Y_i(t))_{i \in \mathbb{N}}$  be a sequence of iid  $\mathcal{S}^\downarrow$ -fragmentation with same semi-group  $(P_S(t), S \in \mathcal{S}^\downarrow, t \geq 0)$ , then, by definition, for all  $n \in \mathbb{N}$  the  $\mathcal{S}^\downarrow$

random variable  $Z^{(n)}(t)$ , obtained by the decreasing rearrangement of the terms  $g_{l_i^{(n)}} \left( Y_1 \left( t(l_i^{(n)})^\alpha \right) \right)$  for  $i$  in  $\mathbb{N}$  :

$$Z^{(n)}(t) = \left( g_{l_1^{(n)}} \left( Y_1 \left( t(l_1^{(n)})^\alpha \right) \right), g_{l_2^{(n)}} \left( Y_2 \left( t(l_2^{(n)})^\alpha \right) \right), \dots \right)^\downarrow$$

has law  $P_{L^{(n)}}(t)$ . In the same way

$$Z(t) = (g_{l_1}(Y_1(t(l_1)^\alpha)), g_{l_2}(Y_2(t(l_2)^\alpha)), \dots)^\downarrow$$

has law  $P_L(t)$ . Now fix  $\epsilon > 0$ , and take  $N > \frac{2}{\epsilon}$ . Then

$$\forall k \geq N, \forall n \in \mathbb{N}, l_k^{(n)} < \epsilon/2.$$

Thus for all  $\omega$

$$\sup_{k \geq N} \left( \text{dist} \left( g_{l_k^{(n)}}(Y_k(t(l_k^{(n)})^\alpha)), g_{l_k}(Y_k(t(l_k)^\alpha)) \right) \right) < \epsilon.$$

On the other hand, by the continuity in probability of the processes  $(Y_i)_{i \in \{1, \dots, N-1\}}$ , we have that almost surely

$$P \left[ \sup_{k \in \{1, \dots, N-1\}} \left( \text{dist} \left( g_{l_k^{(n)}}(Y_k(t(l_k^{(n)})^\alpha)), g_{l_k}(Y_k(t(l_k)^\alpha)) \right) \right) > \epsilon \right] \xrightarrow{n \rightarrow \infty} 0.$$

Thus almost surely, for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$

$$\text{dist} (Z^{(n)}(t), Z(t)) < \epsilon.$$

There is convergence in probability and thus in law.  $\square$

## 2.2 Partition Fragmentations

Most of the results on fragmentation available in the literature are (or can be) formulated in term of a type of fragmentation called *partition fragmentation*, which is basically a process which can be described as a partition of  $\mathbb{N}$  getting finer as time runs.

More precisely, call a subset of  $\mathbb{N}$ , say  $B$ , a "block". When the limit

$$|B| := \lim_{n \rightarrow \infty} \frac{1}{n} \text{Card}\{0 \leq k \leq n : k \in B\}$$

exists, it is called the asymptotic frequency of  $B$ . A partition of  $\mathbb{N}$  can be thought of as a sequence  $B_1, B_2, \dots$  of disjoint blocks whose union is  $\mathbb{N}$ . The labeling obey the following rule : if  $B_i$  is not empty, then its least element is  $i$ . Call  $\mathcal{P}$  the space of the partitions of  $\mathbb{N}$ , and recall that  $\mathcal{P}$  is a metric compact space, see [13].

A finite permutation  $\sigma$  (i.e. a bijection  $\mathbb{N} \rightarrow \mathbb{N}$  such that  $\sigma(n) = n$  for  $n$  large enough) acts on a partition  $\pi$  in the following way : for any  $i$  and  $j$  in  $\mathbb{N}$ ,  $i$  and  $j$  are in the same block of  $\sigma(\pi)$  iff  $\sigma(i)$  and  $\sigma(j)$  are in the same block of  $\pi$ , this equivalence relation can be identified as a partition and thus completely define  $\sigma(\pi)$ .

A measure  $\mu$  on  $\mathcal{P}$  is said *exchangeable* if for any measurable set  $A \subseteq \mathcal{P}$ , for any finite permutation  $\sigma$

$$\mu(A) = \mu(\sigma(A)),$$

where  $\sigma$  acts on the sets in the obvious way.

A  $\mathcal{P}$ -valued process  $\Pi$  is said exchangeable if the permuted process  $\sigma(\Pi)$  has the same distribution as the original process  $\Pi$ . For instance the  $\mathcal{P}$ -valued process  $\Pi_F(t)$  presented in the introduction is exchangeable.

For all  $B \subseteq \mathbb{N}$ , let  $P_B$  be a probability on the partitions of  $B$ . For all  $\pi = (B_1, B_2, \dots) \in \mathcal{P}$ , let  $P_\pi$  be the distribution of the partition with blocks  $B_{(1,1)}, B_{(1,2)}, \dots, B_{(2,1)}, B_{(2,2)}, \dots$  where  $\pi^{(i)} = (B_{(i,1)}, B_{(i,2)}, \dots)$  is a partition of  $B_i$  and has law  $P_{B_i}$ . The family  $(P_\pi, \pi \in \mathcal{P})$  is, in the terminology of Pitman [15], a fragmentation kernel on  $\mathcal{P}$ .

**Définition 4** *Call  $\mathcal{P}$ -fragmentation any exchangeable  $\mathcal{P}$ -valued Markov process, starting from the trivial partition ( $\mathbb{N}$  is the only non empty block), which is continuous in probability and has fragmentation kernels as its transition semi-group.*

We briefly recall some definitions and results on  $\mathcal{P}$ -fragmentations. If  $\pi$  is a random exchangeable partition, by a result of Kingman [13] (see also Aldous [1] for a simpler proof), every block of  $\pi$  has an asymptotic frequency almost surely, i.e.  $|B_i|$  exists with probability 1 for all  $i = 1, \dots$

We call an exchangeable  $\mathcal{P}$ -valued process  $\Pi$  *nice* if with probability 1,  $\Pi(t)$  has asymptotic frequencies for all  $t \geq 0$  simultaneously. Evans and Pitman [11] have shown that it is always the case when  $\Pi$  is an exchangeable  $\mathcal{P}$ -process with proper frequencies (i.e. for each  $t \geq 0$ ,  $\sum_{i \in \mathbb{N}} |B_i(t)| = 1$  almost surely), and Bertoin [5] proved that so-called homogeneous fragmen-

tation were nice. Observe that when  $\Pi(t)$  is nice, the ordered sequence of the asymptotic frequencies is well defined and is a  $\mathcal{S}^\downarrow$ -valued process.

As we shall construct a Markovian semi-group on  $\mathcal{S}^\downarrow$ , we need a notion slightly more general than the asymptotic frequency, well defined for any subset  $B$  of  $\mathbb{N}$ . We write

$$\Lambda(\Pi(t)) = (\Lambda_1(\Pi(t)), \Lambda_2(\Pi(t)), \dots)^\downarrow = (\lambda_1(t), \lambda_2(t), \dots)$$

for the decreasing rearrangement of the quantities

$$\Lambda_i(\Pi(t)) = \liminf_{n \rightarrow \infty} \frac{1}{n} \# \{k \leq n : k \in B_i(t)\}.$$

By extension we also note

$$\Lambda(B) = \liminf_{n \rightarrow \infty} \frac{1}{n} \# \{k \leq n : k \in B\}$$

for any  $B \subseteq \mathbb{N}$ .

$\Lambda$  is a functional of  $\Pi(t)$  that takes its values in  $\mathcal{S}^\downarrow$ . We stress that  $\Lambda$  is not continuous.

Next for every  $C \subseteq \mathbb{N}$  and every  $\pi = \{B_1, B_2, \dots\} \in \mathcal{P}$ , we define the partition of  $C$  induced<sup>1</sup> by  $\pi$  :

$$\pi \cap C = (B_1 \cap C, B_2 \cap C, \dots).$$

**Définition 5** A  $\mathcal{P}$ -fragmentation  $\Pi = (\Pi(t), t \geq 0)$  is called *self-similar with index*  $\alpha \in \mathbb{R}$  if :

1.  $\Pi$  starts a.s. from the trivial partition.
2. The ranked fragmentation  $\Lambda(\Pi)$  associated to  $\Pi$  is continuous in probability.

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<sup>1</sup>there is in fact another natural way of defining this partition : it is to take the image of  $\pi$  by the mapping that sends  $\mathbb{N}$  onto  $C = \{c_1, c_2, \dots\}$  (where  $c_1 < c_2 < \dots$ ) :

$$\pi \circ C = (\{c_j : j \in B_i\}_{i=1, \dots})$$

Suppose now that  $\pi$  is an exchangeable random  $\mathcal{P}$ -valued variable, for all  $k > 0$ , for any finite permutation  $\sigma$  such that

$$\forall i \leq k; \sigma(i) = c_i,$$

$\pi$  and  $\sigma(\pi)$  have same law, thus in the sense of the equality of the finite-dimensional margins  $\pi \circ C$  and  $\pi \cap C$  have same law. Thus in fact any definition could be taken indifferently.

3. For every  $B \subseteq \mathbb{N}$ ,  $\forall t \geq 0$   $P_B(t)$  (in the above notations) is the distribution of  $\Pi(t\Lambda(B)^\alpha) \cap B$ .

When  $\alpha = 0$  we will say that  $\Pi$  is a homogeneous fragmentation.

Following Kingman [13] (see also [1] for a survey), to each  $s = (s_1, s_2, \dots) \in \mathcal{S}^\downarrow$  one can associate a unique exchangeable probability measure  $\mu_s$  on  $\mathcal{P}$  such that  $\mu_s$ -almost every partition has ranked asymptotic frequencies  $s$ .

This is how one proceeds : let  $(X_i)_{i \in \mathbb{N}}$  a family of iid variables such that  $\forall k \in \mathbb{N}$ ,  $P(X_i = k) = s_k$  and  $P(X_i = -i) = 1 - \sum_k s_k$ , then define the  $s$ -paintbox<sup>2</sup> partition ( or " $s$ -paintbox process")  $\Pi$  by the equivalence relation

$$\forall i, j \in \mathbb{N}, i \sim j \Leftrightarrow X_i = X_j.$$

We denote by  $\mu_s$  the law of the  $s$ -paintbox process. It is clear by the LLN that  $\mu_s$ -almost surely  $\Lambda(\Pi) = s$ .

For each self-similar  $\mathcal{P}$ -fragmentation one can take the associated  $\mathcal{S}^\downarrow$  ranked fragmentation, thus defining a map from  $\mathcal{P}$ -fragmentation laws into  $\mathcal{S}^\downarrow$ -fragmentation laws. Suppose now that  $\Pi_1$  and  $\Pi_2$  are two self-similar  $\mathcal{P}$ -fragmentations such that for any fixed  $t$  the  $\mathcal{S}^\downarrow$  variables  $\Lambda(\Pi_1(t))$  and  $\Lambda(\Pi_2(t))$  have same law.  $\Pi_1(t)$  and  $\Pi_2(t)$  being exchangeable, by de Finetti's theorem (see [1]) one can show that they are mixture of paintbox processes directed respectively by  $\Lambda(\Pi_1)$  and  $\Lambda(\Pi_2)$ , i.e.

$$P(\Pi_{1,2} \in A) = \int_{\mathcal{S}^\downarrow} \mu_s(A) P(\Lambda(\Pi_{1,2}) \in ds).$$

We conclude that they have the same distribution. So to every  $\mathcal{P}$ -fragmentation corresponds a different  $\mathcal{S}^\downarrow$ -fragmentation. Our first result will be to show that there is in fact a one to one relation.

## 2.3 From Ranked to Partition Fragmentations

Let  $\Pi$  be a nice self-similar fragmentation of index  $\alpha$ , then it is not difficult to show that its asymptotic frequencies  $\Lambda(\Pi)$  form a self-similar ranked

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<sup>2</sup>The reason for the name (due to Kingman) is the following : imagine that we have a choice of colors  $(c_k)_{k \in \mathbb{N}}$ . Then paint each integer  $n$  independently with a randomly chosen color,  $c_k$  with probability  $s_k$ . Then the partition of  $\mathbb{N}$  defined by the equivalence relation "being of the same color" is the  $s$ -paintbox process.



fragmentation of index  $\alpha$ . Conversely we shall now show that to each  $\mathcal{S}^\downarrow$ -fragmentation  $\lambda$  we can associate a  $\mathcal{P}$ -fragmentation  $\Pi$  with same index of self-similarity such that  $\lambda = \Lambda(\Pi)$ .

**Proposition 6** *We have the following relations between  $\mathcal{S}^\downarrow$  and  $\mathcal{P}$  fragmentations :*

1. *If  $\Pi$  is a  $\mathcal{P}$ -fragmentation then  $\Lambda(\Pi)$  has the finite-dimensional marginal distributions of an  $\mathcal{S}^\downarrow$ -fragmentation. Moreover  $\Lambda$  preserves self-similarity.*
2. *if  $\lambda$  is a  $\mathcal{S}^\downarrow$ -fragmentation, then we can construct  $\Pi_\lambda$  an exchangeable  $\mathcal{P}$ -fragmentation such that  $\Lambda(\Pi_\lambda) \stackrel{\mathcal{L}}{=} \lambda$ . Moreover this construction preserves self-similarity.*

The first point is clear, the difficulty here lies in the second part of this proposition. The main idea is that as  $\mathcal{P}$  is a compact metric space, it is enough to construct an adequate Markovian semi-group to ensure the existence of the desired  $\mathcal{P}$ -process. Then the conservation of the index will be a simple consequence of our construction.

Let  $(P_t(S), t \geq 0, S \in \mathcal{S}^\downarrow)$  be a transition kernel on  $\mathcal{S}^\downarrow$  generated, in the notation of definition (1), by the family  $(P_t(l), t \geq 0, l \in [0, 1])$ .

Let  $\tilde{P}_t(l)$  be the image of  $P_t(l)$  by  $g_{l-1}$  the map  $(x_1, x_2, \dots) \rightarrow (x_1/l, x_2/l, \dots)$ . Let  $(Q_t(l, \cdot), l \in [0, 1], t \geq 0)$  be a family of probability measures on  $\mathcal{P}$  where, for a fixed  $t$ ,  $Q_t(l)$  is a mixture of  $s$ -paintbox processes directed by  $\tilde{P}_t(l)$ , i.e. for  $A \subseteq \mathcal{P}$

$$Q_t(l, A) = \int_{\mathcal{S}^\downarrow} \mu_s(A) \tilde{P}_t(l, ds).$$

For  $B \subseteq \mathbb{N}$  define  $Q_t(B)$  the distribution of  $\Pi_B \cap B$  where  $\Pi_B$  is a  $\mathcal{P}$  valued random variable with law  $Q_t(\Lambda(B))$ . Practically this means that one begins by drawing a variable  $\lambda_B$  with law  $\tilde{P}_t(\Lambda(B))$  and then the  $\lambda_B$ -paintbox partition and then take its intersection with  $B$ .

Now let  $\pi = (\pi_1, \pi_2, \dots) \in \mathcal{P}$  and  $\forall t \geq 0$  let  $(\Pi_{\pi_i}(t))_{i \in \mathbb{N}}$  be a sequence of independent variables with respective law  $Q_t(\pi_i)$ . Define  $Q_t(\pi)$  the law of the partition whose blocks are the blocks of the  $(\Pi_{\pi_i}(t), i \in \mathbb{N})$ .

Our proof of Proposition 6 shall thus consist in showing that the family  $(Q_t(\pi), \pi \in \mathcal{P}, t \geq 0)$  forms a semi-group.

**Proof.** From the above description it should be clear that it suffices to show

$$\forall \pi \in \mathcal{P}, Q_{t+u}(\pi) = \int_{\pi' \in \mathcal{P}} Q_t(\pi') Q_u(\pi, d\pi') \quad (1)$$

in the obvious notation. If for any subset  $B$  of  $\mathbb{N}$  we note  $\mathcal{P}_B$  for the space of the partitions of  $B$ , by construction, (1) is equivalent to

$$\forall B \subseteq \mathbb{N}, Q_{t+u}(B) = \int_{\pi' \in \mathcal{P}_B} Q_t(\pi') Q_u(B, d\pi'). \quad (2)$$

We can reformulate (2) as :  $Q_{t+u}(B)$  is the distribution of the random partition  $\Pi(t, u)$  of  $B$  (and this is what we actually shall prove) obtained by the following two-steps procedure :

1. draw  $\Pi(u) = (\pi_1(u), \pi_2(u), \dots)$  an exchangeable partition of  $B$  with law  $Q_u(B)$ .
2. given  $\Pi(u)$  draw a sequence  $(\Pi_{\pi_i(u)}(t))_{i \in \mathbb{N}}$  of independent  $\mathcal{P}_{\pi_i(u)}$ -variables with respective law  $Q_t(\pi_i(u))_{i \in \mathbb{N}}$ .
3.  $\Pi(t, u)$  is just the collections of all the blocks of the  $\Pi_{\pi_i(u)}(t)$ .

We begin by proving so for  $B = \mathbb{N}$ . By construction we can always suppose that  $\Pi(u)$  is a mixture of paintbox processes directed by  $P_u((1, 0, \dots))$ , i.e. conditionally on  $\lambda(u)$  (a random variable with law  $P_u((1, 0, \dots))$ ),  $\Pi(u)$  is a  $\lambda(u)$ -paintbox process (resp. for each  $i \in \mathbb{N}$   $\Pi_{\pi_i(u)}(t)$  is constructed by taking the intersection of  $\pi_i(u)$  and a  $\lambda^{(i)}$  paintbox-process where  $\lambda^{(i)}$  is a  $\mathcal{S}^\downarrow$ -variable with law  $\tilde{P}_t(|\pi_i(u)|)$ .)

This means that conditionally on  $\lambda(u) = (\lambda_1(u), \lambda_2(u), \dots)$  one draws an i.i.d. sequence of variables  $(X_i)_{i \in \mathbb{N}}$  with values in  $\mathbb{N}$  whose law is  $P(X_1 = k) = \lambda_k(u)$  for any  $k \geq 1$  and  $P(X_i = -i) = 1 - \sum_n \lambda_n(u)$  which determines  $\Pi(u)$  (idem with each  $\Pi_{\pi_i(u)}(t)$  denoting  $(Y_k^{(i)})_{k \in \mathbb{N}}$  the appropriate sequence of variables).

Fix  $\phi$  a bijection from  $\mathbb{N}^2$  in  $\mathbb{N}$  and define the coordinate  $\phi^{-1}(k) = (\alpha_k, \beta_k)$  for all  $k$ , then for  $i$  and  $j$

$$\begin{aligned} i \stackrel{\Pi(u,t)}{\sim} j &\Leftrightarrow \{X_i = X_j \text{ and } Y_i^{(X_i)} = Y_j^{(X_j)}\} \\ &\Leftrightarrow \{(X_i, Y_i^{(X_i)}) = (X_j, Y_j^{(X_j)})\} \\ &\Leftrightarrow \phi(X_i, Y_i^{(X_i)}) = \phi(X_j, Y_j^{(X_j)}) \\ &\Leftrightarrow Z_i = Z_j \end{aligned}$$

where  $Z$  is an obvious notation  $Z_i = \phi(X_i, Y_i^{(X_i)})$ .

Then we have the identity

$$\{Z_i = k\} = \{X_i = \alpha_k, Y_i^{(\alpha_k)} = \beta_k\}$$

As conditionally on  $(\lambda^{(i)}(t))_{i \in \mathbb{N}}$  and  $\lambda(u)$  the  $X_i$  are i.i.d. as well as the sequences  $(Y_k^{(i)})_{k \in \mathbb{N}}$  and are all independent between them, we see that the  $Z_i$  are also i.i.d. As

$$i \stackrel{\Pi(t,u)}{\sim} j \Leftrightarrow \{Z_i = Z_j\}$$

$\Pi(t, u)$  is exchangeable.

The law of an exchangeable random partition is completely determined by the law of its asymptotic frequencies, here the  $\lambda_i(u) \times \lambda^{(i)}(t)$ . As  $\lambda(\cdot)$  is a  $\mathcal{S}^\downarrow$ -fragmentation we have by construction that

$$((\lambda_i(u) \times \lambda^{(i)}(t))_{i \in \mathbb{N}})^\downarrow \stackrel{\mathcal{L}}{=} \lambda(t+u).$$

So  $\Pi(t, u)$  has law  $Q_{t+u}(\{\mathbb{N}\})$ .

Then take  $B$  a subset of  $\mathbb{N}$ . By construction  $Q_{t+u}(B)$  is the law of  $\tilde{\Pi}_{t+u}(B) \cap B$  where  $\tilde{\Pi}_{t+u}(B)$  is a certain  $\mathcal{P}$ -variable and  $\Pi(t, u) = \tilde{\Pi}(t, u) \cap B$  where  $\tilde{\Pi}(t, u)$  is a certain variable. It is clear that replacing the generating family  $(Q_t(l), t \geq 0, l \in [0, 1])$  by  $(Q'_t(l) = Q_t(\Lambda(B)l), t \geq 0, l \in [0, 1])$  the above arguments yields  $\tilde{\Pi}(t, u) \stackrel{\mathcal{L}}{=} \tilde{\Pi}_{t+u}(B)$  and thus for all  $B \subseteq \mathbb{N}$

$$\Pi(t, u) \stackrel{\mathcal{L}}{=} \Pi_{t+u}(B).$$

So we have proved the existence of a Markov  $\mathcal{P}$ -process  $\Pi$  with semi-group  $Q_\pi(t)$ , which, by construction, is a fragmentation whose asymptotic frequencies has same distribution (in the sense of finite-dimensional distributions) as  $\lambda_t$  our starting  $\mathcal{S}^\downarrow$ -fragmentation.

For each ranked fragmentation  $\lambda$  we can thus construct a partition fragmentation  $\Pi_\lambda$  such that  $\Lambda(\Pi_\lambda)$  has same law as  $\lambda$ .

We now turn to the conservation of self-similarity : suppose  $\lambda$  is a self-similar  $\mathcal{S}^\downarrow$ -fragmentation with index  $\alpha$ , so  $\tilde{P}_l(t) = P_1(tl^\alpha)$ , looking at the above construction of the semi-group of  $\Pi_\pi$  shows that  $Q_t(l) = Q_1(tl^\alpha)$ , so  $\Pi$  is also self-similar of index  $\alpha$ .  $\square$

It is now natural to look for some explicit construction of ranked fragmentation i.e. an equivalent of Theorem 1 in [5].

### 3 Homogeneous fragmentation

In [5] J. Bertoin shows how a homogeneous  $\mathcal{P}$ -fragmentation process can be decomposed into a Poisson point process of partitions, whose distribution

is determined by the so-called characteristic measure. We will begin by recalling the facts we need on this topic, and then tackle the analog problem for  $\mathcal{S}^\downarrow$  homogeneous fragmentations.

### 3.1 Lévy-Itô decomposition of homogeneous $\mathcal{P}$ -fragmentations

The distribution of a homogeneous  $\mathcal{P}$ -fragmentation  $\Pi$  is determined by an exchangeable measure  $\kappa$  on  $\mathcal{P}$ , called the characteristic measure of  $\Pi$ , that assign zero mass to the trivial partition and verifies the condition  $\kappa(\mathcal{P}_2^*) < \infty$  where  $\mathcal{P}_2^*$  is the set of the partitions of  $\mathbb{N}$  for which 1 and 2 does not belong to the same block. Given such a measure  $\kappa$ , one can construct an homogeneous  $\mathcal{P}$ -fragmentation admitting  $\kappa$  as its characteristic measure as follows : Let  $K = ((\Delta(t), k(t)), t \geq 0)$  a Poisson point process with values in  $\mathcal{P} \times \mathbb{N}$  with intensity measure  $M := \kappa \otimes \#$  where  $\#$  stands for the counting measure on  $\mathbb{N}$ . This means that for a measurable set  $A \subseteq \mathcal{P} \times \mathbb{N}$  with  $M(A) < \infty$ , the counting process

$$N^A(t) = \text{Card}(s \in [0, t] : (\Delta(s), k(s)) \in A), t \geq 0$$

is a Poisson process with intensity  $M(A)$ , and to disjoint sets correspond independent processes.

Then one can construct a unique  $\mathcal{P}$ -valued process  $\Pi_\kappa = (\Pi_\kappa(t), t \geq 0)$  started from the trivial partition, with càdlàg sample paths, such that  $\Pi_\kappa$  only jumps at time  $t$  at which  $K$  has an atom  $(\Delta(t), k(t))$ , and in that case  $\Pi_\kappa(t)$  is the partition whose blocks are the  $B_i(t_-)$  (the blocks of  $\Pi_\kappa(t_-)$ ) except for  $B_{k(t)}(t_-)$  which is replaced by the partition of  $B_{k(t)}(t_-)$  induced by  $\Delta(t)$  (that is  $\Delta(t) \cap B_{k(t)}(t_-)$ ).

$\Pi_\kappa$  is a homogeneous  $\mathcal{P}$ -fragmentation with characteristic measure  $\kappa$ . Conversely, any homogeneous  $\mathcal{P}$ -valued fragmentation  $\Pi$  has the same law as  $\Pi_\kappa$  for some unique exchangeable measure  $\kappa$ .

As a consequence of Kingman's representation of exchangeable partitions [13], every exchangeable partition measure can be decomposed as the sum of a *dislocation* measure and an *erosion* measure :

- $\delta_\pi$  stands for the Dirac point mass at  $\pi \in \mathcal{P}$ , for all  $n \in \mathbb{N}$  let  $\epsilon_n$  be the partition of  $\mathbb{N}$  with only two non-voids blocks :  $\{n\}$  and  $\mathbb{N} \setminus \{n\}$ , then

for every  $c \geq 0$ , the measure

$$\mu_c = c \sum_{n=1}^{\infty} \delta_{\epsilon_n}$$

is an exchangeable measure. The  $\mu_c$ 's are called *erosion* measures.

- The dislocation measures are constructed from so-called Lévy measures on  $\mathcal{S}^\downarrow$ . We call a measure  $\nu$  on  $\mathcal{S}^\downarrow$  a Lévy measure if  $\nu$  has no atom at  $(1, 0, 0, \dots)$  and verifies the integral condition

$$\int_{\mathcal{S}^\downarrow} (1 - s_1) \nu(ds) < \infty$$

where  $s = (s_1, s_2, \dots)$  denotes a generic sequence in  $\mathcal{S}^\downarrow$ . The mixture of paintbox processes

$$\mu_\nu(\cdot) = \int_{\mathcal{S}^\downarrow} \mu_s(\cdot) \nu(ds)$$

is a measure on  $\mathcal{P}$ , called the *dislocation* measure directed by  $\nu$ .

Then for any  $\kappa$  exchangeable partition measure there exists a unique  $c \geq 0$  and a unique Lévy measure  $\nu$  such that  $\kappa = \mu_c + \mu_\nu$ .

Thus the law of a homogeneous  $\mathcal{P}$ -fragmentation is completely characterized by the pair  $(\nu, c)$ . Using Proposition 6, we conclude that :

**Corollaire 7** *There is a bijective correspondence between the laws of homogeneous ranked fragmentations and the pairs  $(\nu, c)$  where  $\nu$  is a Lévy measure on  $\mathcal{S}^\downarrow$  and  $c \geq 0$ .*

A ranked fragmentation is thus completely characterized (in terms of distribution) by the pair  $(\nu, c)$  associated to its law.

We would like to transfer the Poisson point process construction of  $\mathcal{P}$ -fragmentations to  $\mathcal{S}^\downarrow$ -fragmentations. The main difficulty in doing so comes from the lack of a genealogy structure in this new setting.

To illustrate this, let  $K = (\Delta(t), k(t))$  a PPP with measure intensity  $\mu_\nu \times \#$  and  $\Pi$  the corresponding  $\mathcal{P}$  fragmentation (hence with no erosion), and suppose that at time  $t$  the  $k$ -th block of  $\Pi(t_-)$  (i.e. its least element is  $k$ ) fragments, or otherwise said  $\Lambda(B_k(t_-)) > \Lambda(B_k(t)) > 0$ . Then it is clear that at time  $t$  there is also a dislocation in the associated ranked fragmentation  $\lambda = \Lambda(\Pi)$ . The label of the mass of  $\lambda(t_-)$  that fragments, noted  $\Phi(t_-, k)$ , is an integer that depends on  $\Pi(t_-)$  and  $k$  and can informally be seen as the rank of the size of the  $k(t)$ -th block of  $\Pi(t_-)$ . In the same way that  $\Pi$  is constructed

from  $K$ , one might hope that  $\Lambda(\Pi)$  is constructed from  $(\Lambda(\Delta_t), \Phi(t_-, k_t))$  but we will still have to show that this last point process is a Poisson point process with the right intensity, then that the jump-times of  $\lambda$  are exactly the atom times of  $K$  and finally that  $\lambda$  is a pure-jump process (in a sense to be defined).

But first we show how to get rid of erosion.

### 3.2 Erosion in homogeneous ranked fragmentation

Let us first examine the trivial case when the fragmentation is pure erosion. It is then intuitively clear that the homogeneity in time and space entails that the ranked fragmentation  $\lambda(t)$  with values in  $S^\downarrow$  with characteristics  $(0, c)$  (where the 0 means that the measure  $\nu$  is trivial with mass 0) is given by

$$\lambda(t) = (e^{-ct}, 0, 0, \dots)$$

To demonstrate this define

$$k = \mu_c$$

with  $c > 0$ , and let  $\Pi$  be the  $\mathcal{P}$ -fragmentation associated to the P.P.P.  $K = (\Delta(t), k(t))_{t \geq 0}$  with intensity  $\mu_c \otimes \#$  and values in  $\mathcal{P} \times \mathbb{N}$ .  $\Pi$  can be thought of as an isolation process, indeed at each jump time of  $K$ , say  $t$ , some point of  $\mathbb{N}$ , say  $n$ , is designated, (i.e.  $\Delta(t) = \delta_{\epsilon_n}$ ). If the block containing  $n$ ,  $\beta(n, t)$ , is not reduced to the singleton  $\{n\}$ , then it is fragmented into  $\{n\}$  and  $\beta(n, t) \setminus \{n\}$ , "n is isolated from its block", else nothing happens. Hence, at all time there is only one block which is not a singleton, by an argument that will be established thereafter in Theorem 9, we can always suppose that this block also contains 1. If we consider the restriction of  $\Pi$  to  $\{1, 2, \dots, n\}$ , denoted by  $\Pi^{(n)}$ , then  $\Pi^{(n)}$  only jumps at atom-times of  $K$  for which  $k_t = 1$  and  $\Delta_t \in \{\delta_{\epsilon_1}, \delta_{\epsilon_2}, \dots, \delta_{\epsilon_n}\}$ . The restriction of the Poisson Process to this set is a Poisson Process with intensity of finite mass and have thus discrete jump-times. The processes of the times of exclusion of each point are independent one of the other. By standard calculation on Poisson processes the probability that a given point have been excluded at time  $t$  is  $\exp(-tc)$ , thus the law of the number of point excluded at time  $t$  is a Bernoulli with parameter  $(e^{-ct})$ . By the law of the large number, at time  $t$ , the asymptotic frequency of the only block not reduced to a singleton is  $(e^{-ct})$  almost surely. So a.s. for every  $t \in \mathbb{Q}$

$$\Lambda(\Pi_t) = ((e^{-ct}), 0, 0, \dots)$$

and as  $\lambda_1(t)$  is monotone decreasing the relation holds almost surely for all  $t$ . This result is the key for the following.

**Proposition 8** *If  $\tilde{\lambda}$  is a homogeneous  $(\nu, 0)$  ranked fragmentation, then  $\lambda = (e^{-ct}\tilde{\lambda}(t), t \geq 0)$  is a homogeneous  $(\nu, c)$  ranked fragmentation.*

**Proof.** Let  $\tilde{\Pi}$  and  $\Pi$  be some homogeneous partition fragmentations with characteristics  $(\nu, 0)$  and  $(\nu, c)$  respectively. Then call  $\tilde{\lambda}$  the process of the ordered asymptotic frequencies of  $\tilde{\Pi}$  and  $\lambda$  those of  $\Pi$ . Suppose  $\Pi$  is constructed on the Poisson point process  $K = (\Delta(t), k(t), t \geq 0)$  with characteristic measure  $\mu_\nu + \mu_c$ . Let  $K_1 = (\Delta(t), k(t), t \geq 0)$  the Poisson point process with characteristic measure  $\mu_\nu \otimes \#$  and  $K_2 = (\Delta(t), t \geq 0)$  the Poisson point process with characteristic measure  $\mu_c$ .

Thus  $\Pi$  appears as (i.e. is equal in law to) the intersection of  $\Pi_1$  (constructed on  $K_1$ ) and a pure erosion process  $\Pi_2$  (constructed on  $K_2$ ), i.e.  $\Pi = \Pi_1(\cdot) \cap \Pi_2(\cdot)$  defined by the equivalence relation

$$\forall i, j \in \mathbb{N} : (i \stackrel{\Pi_1(\cdot) \cap \Pi_2(\cdot)}{\sim} j) \Leftrightarrow \left( (i \stackrel{\Pi_1(\cdot)}{\sim} j) \text{ and } (i \stackrel{\Pi_2(\cdot)}{\sim} j) \right).$$

Given a random exchangeable subset of  $\mathbb{N}$ , say  $A$ , independent of  $(\Pi_2(t))_{t \geq 0}$ , with random asymptotic frequency  $l$ , the asymptotic frequency of the subset of  $A$  defined as the points that have not been excluded up to time  $t$  is  $le^{-ct}$  a.s. for all  $t$ .

Therefore  $\Lambda(\Pi(t)) \stackrel{\mathcal{L}}{=} e^{-ct}\Lambda(\Pi_1(t))$ . As we can always suppose that a ranked fragmentation is the associated ranked fragmentation of some partition fragmentation the result is proven.  $\square$

Thus it suffices to know how to construct a homogeneous ranked fragmentation without erosion from a PPP to know how to construct any homogeneous ranked fragmentation.

### 3.3 Construction of homogeneous ranked fragmentation with no erosion

Let  $\lambda$  be an  $\mathcal{S}^\perp$ -fragmentation, with characteristics  $(\nu, c)$ , then for every  $k \in \mathbb{N}$  the process  $\lambda_1(t) + \dots + \lambda_k(t)$  is monotone decreasing.  $\lambda$  is said to be a pure jump process if for any  $k$ ,  $\lambda_1(t) + \dots + \lambda_k(t)$  is a pure jump process.

In the following we shall focus on the case where for each fixed  $t$  there is a infinite number of fragments of strictly positive size almost surely. A

necessary and sufficient condition for this is

$$\nu(s \in \mathcal{S}^\downarrow : s_2 > 0) = \infty.$$

Indeed, fix  $t > 0$  and suppose that  $\lambda_1(t) > 0$ . Then, for any  $\epsilon > 0$ , during the time interval  $[t - \epsilon, t]$ ,  $\lambda_1$  has been affected by an infinite number of dislocation such that at least one small fragment detached from the main one, thus an infinite number of fragments have been created, and the life-time of those variables form a sequence of independent identically distributed random variables, thus with probability one an infinite number of them have survived at time  $t$ . The same line of arguments also shows that  $\inf\{t \geq 0 : \lambda_1(t) = 0\} = \infty$  almost surely.

Although most of the following results are still true for any  $\nu$ , making this hypothesis enables us to focus on the most interesting case and to avoid some technical difficulties.

**Théorème 9** *Let  $\lambda$  be a homogeneous  $\mathcal{S}^\downarrow$ -fragmentation with no erosion ( $c = 0$ ) and Lévy measure  $\nu$  as above (i.e.  $\nu(\{s : s_2 > 0\}) = \infty$ ). Then*

1.  *$\lambda$  is a pure jump process.*
2. *there exists a PPP  $K = (S(t), k(t))_{t \geq 0}$  with values in  $\mathcal{S}^\downarrow \times \mathbb{N}$  and intensity measure  $\nu \otimes \#$ , such that the jumps of  $\lambda$  correspond to the atoms of  $K$ . More precisely,  $\lambda$  only jumps at times at which  $(S(t), k(t))$  has an atom, and at such a time  $\lambda(t)$  is obtained from  $\lambda(t_-)$  by dislocating the  $k(t)$ -th component of  $\lambda(t_-)$  by  $S(t)$  (i.e. replacing  $\lambda_{k(t)}(t_-)$  by the sequence  $\lambda_{k(t)}(t_-)S(t)$ ) and reordering the new sequence of fragments. Conversely if  $(S(t), k(t))$  is an atom then  $\lambda$  has a jump at  $t$ , i.e.  $\lambda_i$  jumps at  $t$  for some  $i$ .*

Although this result is intuitive in regard to the equivalence between  $\mathcal{P}$  and  $\mathcal{S}^\downarrow$  fragmentation, it requires some technical work.

We give ourselves a homogeneous  $(\nu, 0)$   $\mathcal{S}^\downarrow$ -fragmentation  $\lambda$  with  $\nu$  verifying  $\nu(s_2 > 0) = \infty$ . There is no loss of generality in supposing that  $\lambda$  is constructed as follows : Call  $H = ((\Delta(t), k(t)))_{t \geq 0}$  a PPP with measure intensity  $\mu_\nu \otimes \#$  with values in  $\mathcal{P} \times \mathbb{N}$ . Let  $\Pi$  be the homogeneous  $(\nu, 0)$   $\mathcal{P}$ -fragmentation constructed on  $H$ , then define

$$\lambda = \Lambda(\Pi).$$

Call  $\mathcal{F}_t = \sigma\{\Pi_s, s \leq t\}$  the natural filtration of the  $\mathcal{P}$ -fragmentation  $\Pi$ .



Then at any time  $t$ , call  $\phi(t, \cdot) = \phi_t(\cdot)$  the random,  $\mathcal{F}_t$  measurable application from  $\mathbb{N} \rightarrow \mathbb{N} \cup \infty$  (where  $\infty$  serves as a cemetery point) defined as

- if  $|B_k(t)| > 0$  then  $\phi(t, k)$  is the rank of the asymptotic frequency of  $B_k(t)$  (it is well defined because the number of blocks of greater asymptotic frequencies is always finite with an upper bound of  $|B_k(t)|^{-1}$ , and in case two blocks have the same asymptotic frequency, they are ranked as their least element).
- if  $|B_k(t)| = 0$  (with the convention  $|| = 0$ ) then  $\phi(t, k) = \infty$

We also note  $\tilde{k}(t) = \phi(t_-, k(t))$ . Note that under our hypothesis that there is always an infinite number of fragments  $\forall t \geq 0, \mathbb{N} \subset \{\phi(t, k), k \in \mathbb{N}\}$ .

We will first prove that the point process image of  $H$ , noted  $\tilde{K}$ , whose atoms are the points of  $(\Lambda(\Delta(t)), \tilde{k}(t))_{t \geq 0}$  such that  $\tilde{k}(t) \in \mathbb{N}$ , is a Poisson point process with measure intensity  $\nu \otimes \#$ . Then we will show that this is also the process of the jumps of  $\Lambda(\Pi)$  and this last process is a pure jump process so it can wholly be recovered from  $(\Lambda(\Delta(t)), \tilde{k}(t))_{t \geq 0}$ . This will complete the proof of Theorem 9.

**Lemme 10** *The point process  $\tilde{K}(t)$  derived from  $(\Lambda(\Delta(t)), \tilde{k}(t))_{t \geq 0}$  by only keeping the atoms such that  $\tilde{k}(t) \neq \infty$  is a Poisson point process with intensity measure  $\nu \otimes \#$  ;*

**Proof.** Let  $A$  be a subset of  $\mathcal{S}^\downarrow$  such that  $\nu(S) < \infty$ . For  $i = 1, \dots$  let

$$N_A^{(i)}(t) = \#\{u \leq t : \Lambda(\Delta(u)) \in A, k(u) = i\}$$

Then set

$$N_A(t) = \#\{u \leq t : \Lambda(\Delta(u)) \in A, \tilde{k}(u) = 1\}.$$

$N_A(t)$  is increasing, right-continuous with left-limits with jumps of size 1 (the  $N_A^{(i)}(t)$  being independent Poisson processes they do not jump at the same time almost surely). By definition we have

$$dN_A(t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{\phi(t_-, i)=1\}} dN_A^{(i)}(t)$$

Define

$$d\tilde{N}_A^{(i)}(t) = \mathbf{1}_{\{\phi(t_-, i)=1\}} dN_A^{(i)}(t).$$

It is clear that  $\mathbf{1}_{\{\phi(t_-,i)=1\}}$  is adapted and left-continuous in  $(\mathcal{F}_t)$  and hence predictable. The  $N_A^{(i)}(\cdot)$  are i.i.d. Poisson processes with intensity  $\nu(A)$  in  $(\mathcal{F}_t)$ . Thus, for each  $i$  the process

$$M_A^{(i)}(t) = \tilde{N}_A^{(i)}(t) - \nu(A) \int_0^t \mathbf{1}_{\{\phi(u_-,i)=1\}} du = \int_0^t \mathbf{1}_{\{\phi(u_-,i)=1\}} d(N_A^{(i)}(u) - \nu(A)u)$$

is a square integrable martingale. Then define

$$M_A(t) = \sum_{i=1}^{\infty} \int_0^t \mathbf{1}_{\{\phi(u_-,i)=1\}} d(N_A^{(i)}(u) - \nu(A)u)$$

Note  $f_i(t) = \mathbf{1}_{\{\phi(t_-,i)=1\}}$ , then, for all  $i \neq j, \forall t \geq 0$ ,  $f_i(t)f_j(t) = 0$ , and  $\forall t$ ,  $\sum_{i=1}^{\infty} f_i(t) = 1$ .

As the  $N_A^{(i)}(t)$  are independent Poisson processes they do not jump simultaneously and so the martingales  $M_A^{(i)}(t)$  do not either. They are thus orthogonal (see for example chapter 8, Theorem (43)-D in [10] for a proof). Moreover the oblique bracket of  $M$  is

$$\begin{aligned} \langle M_A \rangle(t) &= \sum_{i=1}^{\infty} \left\langle \int_0^t f_i(u) d(N_A^{(i)}(u) - \nu(A)u) \right\rangle \\ &= \nu(A)t \end{aligned}$$

So  $M_A$  is a  $L_2$  martingale.

So we have demonstrated that  $N_A(t)$  is increasing, right-continuous, left limited with jump of size 1 with compensator  $\nu(A)t$ . Using classical results (see for instance chapter 2.6 in [12], Theorem 6.2) we conclude that  $N_A(t)$  is a Poisson process with intensity  $\nu(A)$ . Now take  $B \in \mathcal{S}^\downarrow$  such that  $A \cap B = \emptyset$ , we can use the same construction as above replacing  $A$  with  $B$  and the fact that  $N_A^{(i)}(t)$  and  $N_B^{(i)}(t)$  are independent Poisson processes in the same filtration to see that

$$\{\Lambda(\Delta(u)) : u \geq 0, \tilde{k}(u) = 1\}$$

is a P.P.P. with intensity measure  $\nu$ . the same arguments yield that

$$(\{\Lambda(\Delta(u)) : u \geq 0, \tilde{k}(u) = 2\})$$

is also a P.P.P. with measure intensity  $\nu$ . It is clear that  $N_1$  and  $N_2$  have no jumps in common because the  $N_A^{(i)}(t)$ 's does not, so they are independent. By iteration this show that  $(\Lambda(\Delta(t)), \tilde{k}(t))$  is a P.P.P. with measure intensity  $\nu \times \#$ .  $\square$

Let  $K$  be a P.P.P. on  $\mathcal{P} \times \mathbb{N}$  with intensity measure  $\mu_\nu \otimes \#$  and  $\Pi = (\Pi(t), t \geq 0) = ((B_1(t), B_2(t), \dots), t \geq 0)$  the  $(\nu, 0)$   $\mathcal{P}$ -fragmentation constructed from  $K$ , and define  $\lambda = \Lambda(\Pi) = (\lambda_1(t), \lambda_2(t), \dots)$  the ordered vector of asymptotic frequencies. In the case considered here  $\Pi$  is nice so almost surely for all  $t$   $|B_i(t)|$  exists for all  $i \in \mathbb{N}$ . Recall that  $\phi(t, k)$  is the rank of the asymptotic frequency  $|B_k(t)|$  at time  $t$ .

We now need to show that  $\lambda$  is a pure jump process in the sense that for each  $k$  the decreasing process  $\lambda_1 + \dots + \lambda_k$  is pure jump and that all his jumps are indeed images of some atoms of  $K$  ( $\Lambda$  being not continuous it is not *a priori* evident).

In [5] it is shown that  $|B_1(t)|$ , the asymptotic frequency of the block that contains  $\{1\}$ , is the inverse of the exponential of a subordinator with 0-drift, and so it is a pure-jump process. By the Markov and homogeneity property this implies that for all  $i > 1$  the process  $|B_i(t)|$ , the asymptotic frequency of the block that contains  $i$ , is càdlàg, started at 0, such that at  $\tau_i = \sup\{t \geq 0 : |B_i(t)| = 0\}$  we have  $|B_i(\tau_i)| > 0$  (i.e. it leaves 0 with a jump), and after  $\tau_i$  the process  $\frac{|B_i(t-\tau_i)|}{|B_i(\tau_i)|}$  is the inverse of the exponential of a subordinator with no drift, in particular it is a pure jump process. Furthermore it is clear by construction that all the jumps of  $B_i(\cdot)$  correspond to some atom of  $\tilde{K}$ .

For each  $t$  define  $\psi_t(\cdot)$  the application from  $\mathbb{N} \rightarrow \mathbb{N}$  inverse of  $\phi(t, \cdot)$ , i.e.

$$\psi_t(\phi_t(i)) = i$$

(exists because  $\phi$  is surjective on  $\mathbb{N}$ ).

**Lemme 11** *Under the above assumption on  $\nu$ ,*

- *for all  $k > 0$ ,  $\lambda_1(t) + \lambda_2(t) + \dots + \lambda_k(t)$  is a pure jump process.*
- *with probability one, for all  $t \geq 0$ , if  $t$  is an atom for  $\lambda$  then  $\tilde{K}$  has an atom at  $t$ .*

**Proof.**

We will begin by proving the result for  $\lambda_1$ , the size of the largest fragment and then turn our attention to the small ones.

$\lambda_1$  is a supremum of a countable family of pure jump processes (the  $|B_i(\cdot)|$ ). However it is easy to exhibit an example of a supremum of a countable family of pure jump processes that is not a pure jump process. So the proof will consist in showing that almost surely on a fixed time interval  $\lambda_1$  is the supremum of a finite number of pure jump processes.

For this proof only, it is convenient to work with so-called *interval fragmentations*.

*Interval fragmentations* are a particular case of *object fragmentations* that we presented in the introduction for which the "object"  $E$  is simply the interval  $[0, 1]$  endowed with the Lebesgue measure. More precisely, call  $\nu$  the space of the open subsets of  $[0, 1]$ . Elements of  $\nu$  admit a unique decomposition in intervals (in the sense that the ordered vector of the lengths is unique). An interval decomposition is a process  $F(t)$  with values in  $\nu$  such that for any  $0 \leq s < t$  one has  $F(s) \subseteq F(t)$  i.e.  $F(t)$  is finer than  $F(s)$ .

Take a sequence  $(u_i)_{i \in \mathbb{N}}$  of iid variables uniformly distributed on  $[0, 1]$ .  $F$  is then transformed into a  $\mathcal{P}$ -process  $\Pi$  by the following rule

$$i \stackrel{\Pi(t)}{\sim} j \Leftrightarrow [u_i, u_j] \subseteq F(t).$$

This last process obviously conserves the refinement property, moreover, if we define interval fragmentations to have a scaling and branching property,  $\Pi$  will be a  $\mathcal{P}$ -fragmentation.

We refer to [6] for a precise definition of interval fragmentation and the equivalence between interval fragmentations and partition fragmentations.

There is no loss of generality in supposing that  $\Pi$  is constructed from an interval fragmentation  $F(t)$  and a sequence  $(u_i)_{i \in \mathbb{N}}$  of iid variables uniformly distributed on  $[0, 1]$ .

Denote  $(I_i(t), i \in \mathbb{N})$  the associated ordered length of the interval decomposition of  $F$  (which are also the associated ordered frequencies of  $\Pi(t)$ ). If  $I^{(i)}(t)$  denote the length of the interval that contains  $u_i$  in the interval decomposition of  $F(t)$ , then

$$I^{(i)}(t) = l_i(t)$$

where  $l_i(t) = |\beta(i, t)|$  is the asymptotic frequency of the block of  $\Pi(t)$  that contains  $i$ .

Calling  $\tau_n$  the stopping time  $\inf\{t > 0, |B_n(t)| > 0\}$  we have that at  $\tau_n$

$$\forall i < n, n \stackrel{\Pi(\tau_n)}{\not\sim} i,$$

thus  $u_n$  does not belong to any block of  $F(\tau_n)$  that contains some  $u_i$  for any  $i < n$ , hence the asymptotic frequency of the block of  $F(\tau_n)$  that contains  $u_n$  is bounded from above by  $\sup_{i,j \in \{1, \dots, n\}} |u_i - u_j|$  which converge to 0 almost surely when  $n \rightarrow \infty$ .

Note that

$$\sup_{r \in \mathbb{R}^+} \{|B_n(r)|\} = |B_n(\tau_n)|$$

to see that

$$\lim_{n \rightarrow \infty} \left( \sup_{r > 0} (|B_n(r)|) \right) = 0 \text{ a.s.}$$

Now fix  $\epsilon > 0$  and  $n_0$  and condition on the events  $\{\lambda_1(T) \geq \epsilon\}$ , and

$$\left\{ \sup_{n > n_0} \left\{ \sup_{r > 0} (|B_n(r)|) \right\} \right\} < \epsilon.$$

Note that the probability of the second event can be taken arbitrarily close to 1 by taking  $n_0$  sufficiently large. On this event, for all  $r \in [0, T]$  we have that

$$\lambda_1(r) = \max_{i=1, \dots, n_0} |B_i(r)|.$$

Thus  $\lambda_1(\cdot)$  is a pure jump process because all the  $|B_i(\cdot)|$  are. Moreover  $\lambda_1(\cdot)$  only jumps at times at which  $\tilde{K}$  has an atom for the same reason.

We now turn our attention to the other fragments.

Let  $0 < a < b$  and suppose the result is proven for the  $(\lambda_i, i \in \{1, \dots, k-1\})$ . At time  $a$  there is almost surely an infinite number of blocks each with a positive asymptotic frequency, suppose

$$\lambda_k(a) = |B_{\psi_a(k)}(a)| > \epsilon > 0.$$

Call a child of  $B_{\psi_a(1)}(a)$  a block of  $\Pi(a+u)$ ,  $u > 0$  included in  $B_{\psi_a(1)}(a)$ . Denote by  $C_1(b)$  (resp.  $C_j(b)$ ) the size of the largest child of  $B_{\psi_a(1)}(a)$  at time  $b$  (resp. the size of the largest child of  $B_{\psi_a(j)}(a)$  at time  $b$ ). They are almost surely strictly positive. Let  $\eta > 0$  and condition on the event

$$C_1(b) \wedge C_2(b) \wedge \dots \wedge C_k(b) > \eta$$

then it is clear that  $\eta$  is a lower bound for  $\inf_{t \in [a, b]} \{\lambda_k(t)\}$ , thus the same argument as in the  $\lambda_1$  case allow us to consider only a finite number of fragment to be sure to "catch"  $\lambda_k$ . More specifically, conditioned on the event

$$\left\{ \sup_{n > n_0} \left\{ \sup_{u > 0} \{(|B_n(u)|)\} \right\} < \eta \right\},$$

whose probability can be controlled through  $n_0$  to be as close as we wish to 1, we can write  $\lambda_k$  as

$$\forall u \in [a, b], \lambda_k(u) = \sup \left( \{|B_j(u)|\}_{j=1, \dots, n_0} \setminus \{\lambda_1(u), \lambda_2(u), \dots, \lambda_{k-1}(u)\} \right).$$

As the  $|B_i|$  and the  $\lambda_1, \dots, \lambda_{k-1}$  are pure jump processes,  $\lambda_k$  is a pure jump process on  $[a, b]$ , and its jumps correspond to atoms of  $\tilde{K}$  for which  $\tilde{k}(t) \leq k$  (and these atoms are themselves images of atoms of  $K$  for which  $k(t) \leq n_0$ ). So by induction the result is proven.

□

In conclusion, if we call  $\Gamma$  the set of times at which  $(\Delta(t), k(t))$  has an atom. Then writing  $\lambda(t) = (\lambda_1(t), \lambda_2(t), \dots)$  for  $\Lambda(\Pi(t))$  :

1.  $\lambda(\cdot)$  is a pure jump process, càdlàg and starts almost surely from  $(1, 0, 0, \dots)$
2. if  $t \notin \Gamma$ ,

$$\lambda(t) = \lambda(t_-)$$

3. if  $t$  is a jump-time for  $\lambda$ , then almost surely  $t \in \Gamma$  and  $\lambda(t)$  is the reordering of the concatenation of two sequences :  $(\lambda_i(t_-))_{\{i \neq k(t)\}}$  and  $\lambda_{\phi_t(k(t))}(t_-)\Lambda(\Delta(t))$ .

As  $\lambda$  is a pure jump process it is completely defined by this description.

All we have to do now is collect the preceding results : let  $K = (\Delta(t), k(t))$  be a Poisson point process with measure intensity  $\mu_\nu \otimes \#$  and let  $\Pi$  the associated  $(\nu, 0)$  homogeneous  $\mathcal{P}$ -fragmentation. Then the Poisson point process  $(\Lambda(\Delta(t)), \phi_{t_-}(k(t)))$  and the asymptotic frequency process  $\Lambda(\Pi(t))$  have the desired properties, so Theorem 9 is proved.

## 4 Small time Asymptotic behavior

In this section we use the Poisson construction of ranked fragmentations we just established to study their asymptotic behavior near 0. The results we give are very close in spirit to those concerning the asymptotic behavior of subordinators.

A subordinator, say  $\xi$ , is an increasing Lévy process whose distribution is specified by its Laplace exponent  $\Psi$  that is given by the identity

$$\mathbb{E}(\exp \{-q\xi_t\}) = \exp \{-t\Psi(q)\}$$

and the Lévy-Khintchine formula

$$\Psi(q) = k + dq + \int_{]0, \infty[} (1 - e^{-qx}) \nu(dx)$$

where  $k \geq 0$  is the so-called killing rate,  $d \geq 0$  is the drift coefficient and  $\nu$  a measure on  $]0, \infty[$  with  $\int (1 \wedge x) \nu(dx) < \infty$ , called the Lévy measure of  $\xi$ .

The asymptotic behavior of these processes is well known, for instance we have results concerning their distribution :

$$\frac{1}{t} P(\xi(t) \in \cdot) \xrightarrow{t \rightarrow 0+} \nu(\cdot)$$

(see Corollary 8.9 in [16]).

On the other hand, under conditions of regular variation on the tail of  $\nu$ , there are also results concerning the sample path behavior of the limsup and the law of the iterated logarithm (see for instance the end of chapter III in [4]). More precisely :

- (law of the iterated logarithm) A necessary and sufficient condition for the Laplace exponent  $\Psi$  of  $\xi$  to be regularly varying near  $\infty$  with index  $a \in (0, 1)$  is that the drift coefficient is 0 and  $\bar{\nu}(x) = \nu(]x, \infty[)$  is regularly varying in  $0+$  with index  $-a$ . In this case it holds with probability 1 that

$$\liminf_{t \rightarrow 0+} \left( \frac{\xi(t) \Psi^{-1}(t^{-1} \log |\log t|)}{\log |\log t|} \right) = a(1-a)^{(1-a)/a}.$$

- suppose the drift is 0 and let  $h : [0, \infty) \rightarrow [0, \infty)$  be an increasing function such that the function  $t \rightarrow h(t)/t$  increases as well. Then the following assertions are equivalent :

1. a.s.

$$\limsup_{t \rightarrow 0+} (\xi(t)/h(t)) = \infty;$$

- 2.

$$\int_0^1 \bar{\nu}(h(t)) dt = \infty;$$

- 3.

$$\int_0^1 \{ \Psi(1/h(t)) - (1/h(t)) \Psi'(1/h(t)) \} dt = \infty.$$

Finally if these assertions fail to be true, then almost surely

$$\lim_{t \rightarrow 0+} (\xi(t)/h(t)) = 0.$$

Thus to study the asymptotic behavior of a fragmentation we may benefit from the fact that  $|B_1|$  (the mass of the block that contains 1) can be described in terms of a subordinator (see [5]).

We focus on the behavior of the largest ( $\lambda_1$ ) and of the second block ( $\lambda_2$ ) of a ranked fragmentation event though we have more general result in the case of so-called *binary* fragmentations.

Although the study of  $\lambda_1$  is relatively straightforward,  $\lambda_2$  requires to use some results of the record-processes theory. Most of those that will be used in this section are well known or are adapted from standard facts that can be found in most textbooks on the matter. See [7] for instance.

First note that  $\lambda_2(t)$  is not monotone, more precisely it decreases when the second largest fragment undergoes a dislocation and can increase when the largest fragment undergoes a dislocation and one of the new fragment created becomes the second largest.

The idea is to use the Poisson construction : near 0 the largest fragment is almost of size 1, thus the second largest fragment is always a "direct son" of the main one, and we shall be able to express its law in terms of the distribution of the largest fragment that has detached from the main.

For a general  $\mathbb{R}$ -valued P.P.P.  $K = (K_t, t \geq 0)$  with intensity measure  $\mu$  such that  $\forall \epsilon > 0, \mu([\epsilon, \infty]) < \infty$ , it is possible to define the associated record process  $R(t)$  as follows : at time  $t$

$$R(t) = \max_{s \leq t} \{K_s\}.$$

Let  $\lambda$  be a homogeneous  $\mathcal{S}^\downarrow$  fragmentation with characteristic  $(\nu, c)$  constructed from the P.P.P.

$$K = (S(t), k(t))_{t \geq 0} = ((s_1(t), s_2(t), \dots), k(t))_{t \geq 0}$$

of intensity measure  $\nu \otimes \#$ . Let  $(S^{(i)}(t), t \geq 0) = (s_j^{(i)}(t), j = 1, 2, \dots; t \geq 0)$  be the P.P.P. with values in  $\mathcal{S}^\downarrow$  derived of  $K$  by keeping the points such that  $k(t) = i$  (the second coordinate being always  $i$ , it is not expressed). So  $s_j^{(i)}(t)$  is the relative size of the  $j^{th}$  block of the dislocation occurring at time  $t$  on the  $i^{th}$  block.  $S^{(i)}$  is a P.P.P. with intensity measure  $\nu$ . The  $\mathbb{R}$ -valued point process  $(s_j^{(i)}(t))$  is thus a P.P.P. with intensity

$$\nu_j(dx) = \nu(\{s = (s_1, s_2, \dots) \in \mathcal{S}^\downarrow : s_j \in dx\}).$$



Introduce the function

$$x \rightarrow \bar{\nu}_2(x) = \nu(s \in \mathcal{S}^\downarrow : s_2 \geq x)$$

from  $[0, \frac{1}{2}] \rightarrow \mathbb{R}^+$ , and denote by  $f$  its generalized inverse.

Note that  $\bar{\nu}_2(\cdot)$  is finite, i.e. for all  $x > 0$   $\bar{\nu}_2(x) < \infty$ . To see this, let  $b \in [0, 1/2]$

$$\begin{aligned} \int_{\mathcal{S}^\downarrow} (1 - s_1) \nu(ds) &\geq \int_{\mathcal{S}^\downarrow} s_2 \nu(ds) \\ &= \int_0^{1/2} x \nu_2(dx) \\ &\geq \int_b^{1/2} x \nu_2(dx) \\ &> b \bar{\nu}_2(b) \end{aligned}$$

Let  $R(t)$  designate the record at time  $t$  of the P.P.P.  $s_2^{(1)}(\cdot)$  which is well defined according to the above argument.

**Proposition 12** *Let*

$$\lambda = (\lambda(t), t \geq 0) = (\lambda_1(t), \lambda_2(t), \dots), t \geq 0)$$

*be a homogeneous  $\mathcal{S}^\downarrow$  fragmentation with characteristic  $(\nu, c)$ , then*

1. *there exists a subordinator  $\xi$  with drift  $c$  and Lévy measure*

$$L(dx) = e^{-x} \nu(-\log s_1 \in dx), x \in ]0, \infty[$$

*such that*

$$1 - \lambda_1(t) = 1 - \exp \xi(t)$$

*for  $t$  small enough a.s.*

- 2.

$$\lambda_2(t) \sim R(t), \quad t \rightarrow 0 + \quad a.s.$$

**Proof.** (*Proposition 12-(1)*) Assume that  $c = 0$ , then consider  $\tilde{\nu}$  the image of  $\nu$  by the application  $\mathcal{S}^\downarrow \rightarrow \mathcal{S}^\downarrow : (s_1, s_2, \dots) \rightarrow (s_1, 0, \dots)$ . Let  $\tilde{\lambda}$  a homogeneous  $(\tilde{\nu}, 0)$   $\mathcal{S}^\downarrow$ -fragmentation, which thus has no erosion and almost surely for all  $t$  only one block that has positive mass. There is no loss of generality in supposing  $\tilde{\lambda} = \Lambda(\tilde{\Pi})$  where  $\tilde{\Pi} = (\tilde{B}_1, \tilde{B}_2, \dots)$  is a homogeneous  $(\tilde{\nu}, 0)$

$\mathcal{P}$ -fragmentation. Define  $\tilde{\xi}(t) = -\log(\tilde{B}_1(t))$ . As long as  $\tilde{\xi}(t) < \infty$  we have that  $\xi(t) = -\log(\tilde{\lambda}_1(t))$  (because it is the only block which is not reduced to a singleton).

Next we condition on the event  $\lambda_1(t) \geq 1/2$ , for any  $s \leq t$ ,  $\lambda_1(s)$  is either  $\lambda_1(s_-)$ , or the largest fragment issued from a dislocation of  $\lambda_1(s_-)$ . By right continuity  $P(\lambda_1(t) \geq 1/2) \xrightarrow[t \searrow 0]{} 1$ . On  $\lambda_1(t) \geq 1/2$  one has that  $\forall s \leq t$ ,  $\lambda_1(s) = \tilde{\lambda}_1(s)$  (one can construct  $\lambda$  and  $\tilde{\lambda}$  from  $K$  (the same PPP) using its image by the above mentioned transformation for  $\tilde{\lambda}$ ). Thus conditionally on  $s \leq t$

$$1 - \lambda_1(s) = 1 - \exp(-\tilde{\xi}(s)).$$

□

This equivalence relation combined with subordinator properties have immediate consequences such as

$$\frac{1}{t}P(1 - \lambda_1(t) > x) \xrightarrow[t \searrow 0]{} L([-\log(1 - x), 1])$$

where  $L(dx) = e^{-x}\nu(-\log(s_1) \in dx)$ .

For the second point the idea is to describe the asymptotic behavior of  $\lambda_2$  in terms of the records of  $s_2^{(1)}$ . We begin with the following technical lemma

**Lemme 13** *Let*

$$\chi_t = \left( \prod_{u \in [0, t[} s_1^{(1)}(u) \right) \left( \prod_{u \in [0, t[} s_1^{(2)}(u) \right),$$

and suppose  $c = 0$  (there is no erosion) then on the event  $\{\lambda_1(t) \geq 1/2\}$

$$\chi_t R(t) \leq \lambda_2(t) \leq R(t).$$

**Proof.** As noted earlier, one can suppose that  $\lambda(\cdot)$  is the asymptotic frequency of some  $(\nu, 0)$   $\mathcal{P}$ -fragmentation  $\Pi$ , and  $K$  is the image of the P.P.P.

$$(\Delta(\cdot), k(\cdot)) \rightarrow (\Lambda(\Delta(\cdot)), \phi(\cdot, k(\cdot)))$$

with intensity measure  $(\mu_\nu \otimes \#)$ . At time  $t$  we recall the notation  $\psi_t(1)$  for the least element of the block of greatest asymptotic frequency in  $\Pi$ , which is well defined.

Fix  $t$ , and consider  $(\Pi(t-u))_{u \in [0,t]}$ , the fragmentation where the time have been reversed. Informally it is a coalescence, whose final state at  $u = t$  is almost surely the trivial partition and which is left continuous. Thus the functional

$$f_i(u) = \mathbf{1}_{\{i \stackrel{\Pi(t-u)}{\sim} \psi_t(1)\}}$$

(that is  $f_i(u)$  is 1 if at time  $(t-u)$  if  $i$  is in the same block that the integer which is the least element of the largest block at time  $t$  and 0 otherwise) is left continuous and  $f_i(t) = 1$  a.s. Thus, almost surely

$$D_i(t) = t - \sup\{u \in [0, t] : f_i(u) = 0\} < t.$$

Note that as we are on  $\{\lambda_1(t) \geq \frac{1}{2}\}$ ,  $\psi_t(1)$  is always in the block of greatest asymptotic frequency of  $\Pi(t-u)$  for any  $u \in [0, t]$ ; so  $D_i(t)$  is the *detachment time* of  $i$  from the main block (if  $i$  is still in the main block,  $D_i(t)$  is taken equal to  $t$ ).

Now take  $k \geq 2$ , and suppose that at time  $t$  there is at least  $k$  blocks (almost surely the case under our hypothesis) so  $\psi_t(k)$  (the least element of the block of  $k$ -th greatest asymptotic frequency in  $\Pi$  at time  $t$ ) is well defined, almost surely

$$D_{\psi_t(k)}(t) > 0.$$

so if we note  $\beta(i, u)$  for the block of  $\Pi(u)$  that contains  $i$  and  $D(k, t) = D_{\psi_t(k)}(t)$  we have that

$$\begin{aligned} \beta((\psi_t(k)), D(k, t)_-) &= \beta(\Psi_{D(k, t)_-}(1), D(k, t)_-) \\ |\beta((\psi_t(k)), D(k, t))| &< |\beta(\Psi_{D(k, t)}(1), D(k, t))| \end{aligned}$$

(recall that  $\beta(\Psi_t(1), t)$  is the largest block at time  $t$ ). Thus  $\lambda_k(t) \leq s_2^{(1)}(D_{\psi_t(k)}(t))$ . As obviously

$$s_2^{(1)}(D_{\psi_t(k)}(t)) \leq R(t)$$

we conclude that

$$\lambda_k(t) \leq R(t). \tag{3}$$

We now prove the lower-bound part of the lemma.

Let  $T(t) = \inf\{u \leq t : R(u) = R(t)\}$  (the "record-time"). Note that for all  $u \in [0, t]$  at which  $S^{(2)}$  has an atom,

$$\lambda_2(u_-)s_1^{(2)}(u) \leq \lambda_2(u),$$

this is not an equality because the largest fragment issued of the dislocation of  $\lambda_2(u_-)$  can be smaller than  $\lambda_3(u_-)$ . Then, for all  $u \in [0, t]$  not an atom for  $S^{(2)}$ ,

$$\lambda_2(u-) \leq \lambda_2(u),$$

this is due to the fact that  $u$  could be an atom for  $S^{(1)}$ , for which  $\lambda_1(u_-)s_2^{(1)}(u) > \lambda_2(u_-)$ . Recalling that we are still conditioning on  $\{\lambda_1(t) > \frac{1}{2}\}$  we have, using the fact that  $\lambda_2$  is a pure jump process, that

$$\lambda_2(T(t)) \left( \prod_{u \in [T(t), t[} s_1^{(2)}(u) \right) \leq \lambda_2(t) \quad (4)$$

and here again this is not an equality because a reordering might occurs.

Then remark

$$\lambda_2(T(t)) \geq R(t) \left( \prod_{u \in [0, T(t)[} s_1^{(1)}(u) \right). \quad (5)$$

In words : at the time of the record  $R(1, t)$ , the second fragment issued of the dislocation of  $\lambda_1$ , is not necessarily  $\lambda_2$ , but in any case it is smaller or equal.

We can combine (4) and (5) to get

$$\chi_t R(t) \leq \lambda_t(2) \quad (6)$$

□

We can now prove the second part of proposition 12 :

**Proof.** *Proposition 12-(2)* When  $c = 0$  we now only have to show that  $\chi_t \xrightarrow[t \searrow 0]{} 1$  almost surely.  $\left( \prod_{u \in [0, t[} (s_1^{(1)}(u)) \right)$  and  $\left( \prod_{u \in [0, t[} (s_1^{(2)}(u)) \right)$  are independent and identically distributed, and on the event  $\left( \prod_{u \in [0, t[} (s_1^{(1)}(u)) \right) \geq \frac{1}{2}$  this last quantity is exactly the  $\lambda_1(t)$  of some  $(\nu, 0)$  fragmentation, thus almost surely

$$\left( \prod_{u \in [0, t[} (s_1^{(1)}(u)) \right) \xrightarrow[t \searrow 0]{} 1$$

which thus concludes our proof.

Finally in the case of a homogeneous  $(\nu, c)$  fragmentation  $\lambda$  with  $c \geq 0$ , the effect of the erosion is just of multiplying the size of each fragment by a factor  $e^{-ct}$ . So clearly the upper bound of Lemma 13 is still valid, on the other hand we have

$$\xi(t)e^{-ct}R(t) \leq \lambda_2(t)$$

and only a slight modification of the proof for the case  $c = 0$  is needed.  $\square$

**Remarque 1** *If  $\bar{\nu}_2(\cdot)$  is regularly varying with index  $(-a)$  in  $0^+$ ,  $a \geq 0$ , classical results of record-processes theory used with proposition 12 show that*

$$\frac{\lambda_2(t)}{f(\frac{1}{t})} \xrightarrow{\mathcal{L}} L$$

when  $t \searrow 0$  where  $L$  is the extreme law with distribution function  $\exp(-x^{-a})$ .

**Remarque 2** *Suppose that  $\lambda$  is a binary fragmentation, that is  $\nu$  has its support in the subset of  $\mathcal{S}^\downarrow$  defined as  $\{s \in \mathcal{S}^\downarrow, s_3 = s_4 = \dots = 0\}$  and that  $\bar{\nu}_2(x) = \nu(\{s \in \mathcal{S}^\downarrow : s_2 \geq x\})$  is regularly varying near 0 with index  $-a$ . Then using the same ideas as in the above arguments one can show that we have the following asymptotic distributions of the renormalized  $\lambda_k$  for any  $k > 1$  :*

$$\forall k > 1, a.s. \quad \lambda_k(t) \underset{t \searrow 0^+}{\sim} R_2(k, t).$$

As a consequence

$$\frac{\lambda_k(t)}{f(1/t)} \xrightarrow{\mathcal{L}} L(k, a)$$

where  $L(k, a)$  is the law with repartition function

$$F_{k,a}(x) = \left( \sum_{i \in [0, k-1]} e^{-x^{-a}} \frac{(x^{-ai})}{i!} \right)$$

and  $f$  is the generalized inverse of  $x \rightarrow \bar{\nu}_2(x)$ . More generally, the convergence in law holds jointly, the limit distribution function for the  $N$  largest blocks being given by

$$f_N(x_2, x_3, \dots, x_N) = \left( \prod_{i=2}^{i=N-1} (\exp -x_i^{-a}) \right) \int_0^{x_N} (\exp -u^{-a}) \nu(s_2 \in du)$$

for  $x_1 > x_2 > \dots > x_N$  (see [14] for instance).

**Remarque 3** *In the case where the fragmentations considered are not homogeneous but only self-similar and without erosion, a slightly more technical version of Theorem 9 still stands : i.e. it is possible to give an explicit Poisson construction of any  $(\alpha, \nu, 0)$   $\mathcal{S}^\downarrow$ -fragmentation. This allow us to extend the results of section 4 to the case of a self-similar fragmentation with index  $\alpha > 0$ .*

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